Infinite Series: A Compact Reference

Compiled by Damon Scott

Table 1: Basic Tests for Convergence

Name	When to use	Hypotheses	What you do	What you conclude
Geometric Series Test	(**a**) You see a geometric series, one where each term is some *fixed* multiple of the term before it. (**b**) You see the *base* is fixed, and the *exponent* is the index of summation.	Series must be geometric.	Manipulate $\sum_{n=1}^{\infty} a_n$ until it looks like $\sum_{n=1}^{\infty} r^n$ where r is fixed with respect to n.	• $r \in (-1,1) \Rightarrow \sum_{n=1}^{\infty} a_n$ converges. • $r \notin (-1,1) \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges
Fixed Power Series Test a.k.a. **The *p*-Series Test**	(**a**) You see a fixed power series, one of the form $\sum 1/(n^p)$. (**b**) You see the *exponent* is fixed, and the *base* is the index of summation.	Series must be a fixed power series.	Manipulate $\sum_{n=1}^{\infty} a_n$ until it looks like $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is fixed with respect to n.	• $p \in (-\infty,1] \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges. • $p \in (1,\infty) \Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
The n^{th} Term Test	(**a**) Terms are not going to zero: the series isn't even *trying* to converge. (**b**) Terms contain unusually strange or bizarre components. (**c**) Alternating Series Test failed.	None.	Find $\lim\limits_{n\to\infty} a_n$. Alternatively, find $\lim\limits_{n\to\infty} \lvert a_n \rvert$ and reach the same conclusions.	• $\lim a_n = 0 \Rightarrow$ Test fails (!) • $\lim a_n \neq 0 \Rightarrow \sum a_n$ div. • $\lim a_n$ does not exist $\Rightarrow \sum a_n$ diverges.
Simple Comparison Test	(**a**) You see an inequality going the right way. (**b**) You see sin n or cos n as a factor (must couple with Absolute Convergence Test.)	Everything must be positive.	Choose, as test series, $\sum_{n=1}^{\infty} b_n$. Show $a_n \leq b_n$ for all large n, or show $a_n \geq b_n$ for all large n. Find conv. or div. of $\sum_{n=1}^{\infty} b_n$	• $\{a_n \leq b_n$ and $\sum b_n$ conv.$\}$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. • $\{a_n \geq b_n$ and $\sum b_n$ div.$\}$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges. • Other cases $\Rightarrow$ Test fails.
Limit Comparison Test	(**a**) The sequence of terms, a_n, is a rational function of n. (**b**) There are nuissance terms that you do not believe will affect convergence.	Everything must be positive.	Choose, as test series, $\sum_{n=1}^{\infty} b_n$. (Choose b_n as equal to a_n but without the "nuissance" terms). Find $L = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$. Find conv. or div. of $\sum b_n$.	• $L \in (0,\infty) \Rightarrow \sum a_n$ and $\sum b_n$ conv. or div. together. • $L = 0 \Rightarrow a_n \leq b_n$: conclude as for Simple Comparison Test. • $L = \infty \Rightarrow a_n \geq b_n$: conclude as for Simple Comparison Test

Table 2: More Tests for Convergence

Name	When to use	Hypotheses	What you do	What you conclude
Integral Test	(**a**) The corresponding function is easy (or not too hard) to integrate. (**b**) To prove convergence or divergence of some famous series.	Terms must be positive and decreasing.	Choose $f(x)$ so that $f(n) = a_n$ for all large n. Check that f is positive and decreasing. Find conv. or div. of $\int_1^\infty f(x)\,dx$. (May start integral after 1 with same results.)	• f passes check $\Rightarrow$ $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^\infty a_n$ conv. or div. together. • f doesn't pass check $\Rightarrow$ Test fails.
Alternating Series Test	(**a**) You see $(-1)^n$ or $(-1)^{n+1}$ as a factor. (**b**) You see $\sin(\pi n)$ or $\cos(\pi n)$ as a factor. (**c**) Absolute Convergence Test failed.	None.	Answer: * Is $\{a_n\}$ alternating in sign? * Does $\lvert a_n\rvert \to 0$ as $n \to \infty$? * Is $\{\lvert a_n\rvert\}$ a decreasing function of n?	• 3 Yesses $\Rightarrow$ $\sum_{n=1}^\infty a_n$ converges. • 0, 1 or 2 Yesses $\Rightarrow$ Test fails.
Absolute Convergence Test	(**a**) When you would like all the terms to be positive to satisfy the hypotheses of another test.	None.	Find conv. or div. of $\sum_{n=1}^\infty \lvert a_n\rvert$ *Note:* Do <u>not</u> find conv. or div of $\sum_{n=1}^\infty a_n$	• $\Sigma \lvert a_n\rvert$ conv. $\Rightarrow$ Σa_n conv. • $\Sigma \lvert a_n\rvert$ div. $\Rightarrow$ Test fails.
Absolute versus Conditional Convergence Test	(**a**) To show whether the convergence, if any, is absolute or conditional.	None.	Find conv. or div. of $\sum_{n=1}^\infty \lvert a_n\rvert$. If that diverges, then go on and determine conv. or div. of $\sum_{n=1}^\infty a_n$. *Alternatively,* find conv. or div. of $\sum_{n=1}^\infty a_n$. If it converges, then go on a determine conv or div. of $\Sigma \lvert a_n\rvert$.	• $\Sigma \lvert a_n\rvert$ conv $\Rightarrow$ Σa_n converges absolutely • ($\Sigma \lvert a_n\rvert$ div. and Σa_n conv.) $\Rightarrow$ Σa_n converges conditionally. • Σa_n div. $\Rightarrow$ Σa_n diverges. • ($\Sigma \lvert a_n\rvert$ conv. and Σa_n div.) $\Rightarrow$ cannot happen.
Ratio Test	(**a**) Exponentials or factorials (or both) are dominating. (**b**) All "radius of convergence" problems.	None.	Find $r = \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$	• $r \in (-1,1) \Rightarrow$ $\sum a_n$ conv. absolutely. • $r \in (-\infty,-1) \cup (1,\infty) \Rightarrow$ $\sum a_n$ diverges. • $r = -1$ or $r = +1$ $\Rightarrow$ Test fails.
Root Test	(**a**) Terms appear to beg to have their n^{th} roots taken	Terms must be positive.	Find $r = \lim_{n\to\infty} \sqrt[n]{a_n}$	Conclude as for Ratio Test.

Table 3: Advanced Tests for Convergence

Name	When to use	Hypotheses	What you do	What you conclude
Definition of Convergence	**(a)** To prove from first principles. **(b)** Sum telescopes. ***Note*:** Only the difficulty of the "What you do" part prevents this from being a universal test. There are no hypotheses and no "Test fails" component in the conclusion. It is, after all, the very *definition* of convergence.	None.	Define $b_n = \sum_{i=1}^{n} a_i$. Somehow find and simplify b_n. Find $\lim_{n\to\infty} b_n$. Notice whether this limit exists as a real number. (Note that 0 is a real number but "∞" is not. The limit might also oscillate or otherwise fail to exist completely.)	• $\lim_{n\to\infty} b_n$ exists as a real number $\Rightarrow \sum_{i=1}^{\infty} a_i$ converges to the same number, $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges to the same number. • Otherwise $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.
Cauchy Condensation Test	**(a)** You want to remove logarithms. **(b)** Just for fun.	Terms must be positive and decreasing.	Find the convergence or divergence of $\sum_{n=1}^{\infty} 2^n \cdot a_{(2\wedge n)}$	• $\sum_{n=1}^{\infty} 2^n \cdot a_{(2\wedge n)}$ and $\sum_{n=1}^{\infty} a_n$ conv. or div. together.
Ratio Comparison Test 1	**(a)** As an act of desperation. **(b)** To prove some other tests.	Everything must be positive.	Form, as test series, $\sum_{n=1}^{\infty} b_n$. Find $L_1 = \lim_{n\to\infty} \dfrac{(a_{n+1}/a_n)}{(b_{n+1}/b_n)}$. Find conv. or div. of $\sum_{n=1}^{\infty} b_n$.	• $L_1 > 1$ and $\sum b_n$ div $\Rightarrow \sum a_n$ div. • $L_1 < 1$ and $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv. • Other cases $\Rightarrow$ Test fails.
Ratio Comparison Test 2	**(a)** As an act of desperation.	Everything must be positive.	Form, as test series, $\sum_{n=1}^{\infty} b_n$. Find $L_2 = \lim_{n\to\infty} \left(\dfrac{a_{n+1}}{a_n} - \dfrac{b_{n+1}}{b_n} \right)$. Find conv. or div. of $\sum_{n=1}^{\infty} b_n$.	• $L_2 > 0$ and $\sum b_n$ div $\Rightarrow \sum a_n$ div. • $L_2 < 0$ and $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv. • Other cases $\Rightarrow$ Test fails.
First Order Raabe's Test	**(a)** Ratio or Root Test failed unexpectedly. **(b)** Series is one of those named hypergeometric.	Terms must be positive.	Find $L = \lim_{n\to\infty} \left(\dfrac{n a_n}{a_{n+1}} - n \right)$.	• $L > 1 \Rightarrow \sum a_n$ conv. • $L < 1 \Rightarrow \sum a_n$ div. • $L = 1 \Rightarrow$ Test fails. (*Nota bene:* the conclusion is *reversed* from that for the Ratio Test.)

Table 4: Spectrum of Growing Functions

All functions displayed on the spectrum go to infinity as n goes to infinity.

$$\ln(\ln(n)) \quad \| \quad \ln(n) \quad \| \quad n^{0.01} \quad n^{1/2} = \sqrt{n} \quad n^{0.6} \quad n \quad n^2 \quad n^{3.8} \quad n^{5325} \quad \| \quad (1.0001)^n \quad (1.5)^n \quad 2^n \quad e^n \quad 327^n \quad n! \quad n^n \quad \| \quad e^{(e\wedge n)} \quad (n!)!$$

Table 5: Some Numerical Methods for Evaluating or Estimating the Sums of Convergent Series

Name	Hypotheses	Sum being estimated		Best estimate for the sum	Range of error	Notes
Finite Geometric	r must be fixed with respect to n.	$\sum_{n=a}^{b} r^n$	$=$	$\dfrac{r^a - r^{b+1}}{1-r}$	± 0	*Mnemonic:* The sum of a geometric series is the first term minus the after-last term all over one minus the ratio.
Infinite Geometric	r must be fixed w.r.t. n and $\lvert r\rvert < 1$.	$\sum_{n=a}^{\infty} r^n$	$=$	$\dfrac{r^a}{1-r}$	± 0	*Mnemonic:* As above, where "the after-last term" is taken to be $\lim_{n\to\infty} r^{n+1}$.
Telescoping	Series telescopes	$\sum_{n=a}^{\infty} (b_n - b_{n+1})$	$=$	$b_a - \lim_{n\to\infty}(b_n)$	± 0	Covers one case only; finding these sums in general is a *technique* not a *formula*.
Stabilizing Digits	Series converges by the Ratio Test, preferably with $\lvert r\rvert$ not too close to 1.	$\sum_{n=1}^{\infty} a_n$		Compute partial sums, and place partial sums in a column. Look and see which digits have stabilized. Those digits are good.	$\pm 10^{-m}$	where m is the number of digits that have stabilized. Do not use this method unless the hypotheses are satisfied!
Integral	Series converges by the Integral Test.	$\sum_{n=1}^{\infty} a_n$	$=$	$\sum_{n=1}^{m-1} a_n + \int_m^{\infty} f(x)\,dx + \dfrac{a_m}{2}$	$\pm \dfrac{a_m}{2}$	In practice, the error is usually much lower.
Alternating	Series converges by the A.S.T.	$\sum_{n=1}^{\infty} a_n$	$=$	$\sum_{n=1}^{m-1} a_n + \dfrac{a_m}{2}$	$\pm \dfrac{\lvert a_m\rvert}{2}$	Be very careful with the signs, plus or minus, in applying this formula.

Table 6: Some Famous Series

The Harmonic Series, $\sum_{n=1}^{\infty} 1/n$, diverges, slowly, to infinity, as is shown by the Integral Test or by Cauchy Condensation.

The Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n+1}/n$, converges by the Alternating Series Test; it is known to converge slowly to ln 2.

The Zeno Series, $\sum_{n=1}^{\infty} (1/2)^n$, converges to 1 by the Definition of Convergence. **The Zero Series**, $\sum_{n=1}^{\infty} 0$, converges to zero.

The Infinite Accumulation of a Constant is the series $\sum_{n=1}^{\infty} 1$, which diverges to infinity by the Definition of Convergence.

Various First-Logarithmic Series, $\sum_{n=2}^{\infty} 1/(n \cdot (\ln(n))^p)$ for various fixed real numbers p, converge for $p > 1$ and diverge otherwise, as is shown by the Integral Test.